第一部份 微分方程 Differential Equations (50%)

1. (a) Solve the following equation \( 2xy \frac{dy}{dx} + 2y^2 = 3x - 6 \) (8%)

(b) Solve the following equation \( \frac{d^2x}{dx^2} + 4x = 3e^{ix} + 7 \cos(t) \) (7%)

(c) Solve the following partial differential equation (10%)

\[
\frac{\partial^2 u}{\partial x^2} = 4 \frac{\partial^2 u}{\partial t^2}
\]

subject to boundary condition
\( u(0,t) = u(2,t) = 0 \)

and initial condition
\[
\begin{cases}
  u(x,0) = 0 & 0 < x < 2 \\
  u_t(x,0) = f(x) & 0 < x < 2
\end{cases}
\]
2. Figure 1 shows a simplified model of a car riding over a road surface.
   (a) Derive the differential equation relating road profile $x_t$ to car body displacement $x_c$. (6%)
   (b) If $x_c(t) = \sin(t)$, find the general solution of $x_c$ by the method of undetermined coefficients. (6%)

If we were to simulate the case of the car riding over a bump (see Figure 2), we can model the bump as half of a sine wave during the time interval $[a, b]$ and with the amplitude $A$.
   (c) Derive $x_c(t)$ using step functions. (6%)
   (d) Assuming zero initial conditions, find the Laplace transform of $x_c(t)$. (7%)

![Fig. 1](image1)

![Fig. 2](image2)

第二部份 練性代數 Linear Algebra (50%)

1. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$ (10%)

2. Given $A = \begin{bmatrix} 6 & -10 \\ 3 & -5 \end{bmatrix}$ find $A^{10}$ by the diagonalization method. (10%)

3. Prove that the eigenvalues of a Hermitian matrix are real. (10%)

4. Given a set of data points (1,1), (2,2), (3,4), (4,6), and (5,5), determine the equation of a best-fit line by the method of least squares. (10%)

5. Find out what type of conic section the following quadratic form represents and transform it to principal axes:

$$3x_1^2 + 4\sqrt{3}x_1x_2 + 7x_2^2 = 9$$
第三部 份 複 變 (Complex Variables) 50%

1. If $z$ is a complex variable, evaluate each of the following using theorems of limits.
   (a) $\lim_{z \to 2i} \frac{(2z+3)(z-1)}{z^2 - 2z + 4} = ?$ (5%)
   $\lim_{z \to 2e^{i\pi}} \frac{z^3 + 8}{z^4 + 4z^3 + 16} = ?$ (7%)

2. $f(z) = \frac{z^4 + z^3 + 2}{(z-1)(3z+2)^2}$
   (a) Locate and name all singularities of $f(z)$. (5%)
   (b) Determine where $f(z)$ is analytic. (3%)

3. Complex integration
   (a) If $C$ is defined by $\pi y = x^2$, evaluate $\int_C (z+2)e^z \, dz$ from $(0,0)$ to $(\pi,1)$. (7%)
   (b) If $f(z)$ is analytic inside and on the boundary $C$ of a simply connected region $R$, evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} \, dz$. (8%)

4. Evaluate $\frac{1}{2\pi i} \int_{\gamma} \frac{e^z}{\sqrt{z+1}} \, dz$ where $a$ and $t$ are any positive constants. (15%)